Bayesian Bradley-Terry Modeling with Multiple Game Outcomes with Applications to European and College Hockey

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Bradley-Terry Model

- Bradley-Terry model (Bradley and Terry 1952) assigns a probability of team i beating team j as

$$\theta_{ij}^W = \frac{\pi_i}{\pi_i + \pi_j} = \frac{1}{1 + e^{\lambda_i - \lambda_j}}$$

- π_i is the strength of team i, π_j is the strength of team j, $\lambda_i = \ln \pi_i$, and $\lambda_j = \ln \pi_i$
- Inference problem is to say something about $\{\pi_i\}$, or equivalently $\{\lambda_i\}$ given D, or equivalently n_{ij}^W and n_{ij}^L.

Gaussian Approximation

- The gaussian approximation with c as the constant is:

$$p(\{\lambda_k\}|D) \approx \mathrm{c} \times \exp\left(-\frac{1}{2}\sum_{i=1}^{t}\sum_{j=1}^{t}(\lambda_i - \widehat{\lambda}_i)H_{ij}(\lambda_j - \widehat{\lambda}_j)\right)$$

- $\{H_i j\}$ works like an inverse of the variance-covariance matrix of a multivariate Gaussian (normal) approximation to the pdf for λ_i

$$H_{ij} = \delta_{ij}\sum_{k=1}^{t} n_{ik}\hat{\theta}_{ik}^W \hat{\theta}_{ki}^W - n_{ij}\hat{\theta}_{ij}^W \hat{\theta}_{ji}^W$$

Standard Bradley-Terry Model for ECAC

- This section uses the 2020-2021 ECAC Hockey season, with four colleges (Clarkson, Colgate, Quinnipiac, and St. Lawrence)
- Example of an unbalanced schedule, since the last four Clarkson vs. St. Lawrence games were canceled when Clarkson ended their season early

Maximum Likelihood Estimate

- Maximum Likelihood Estimate (MLE) generates expected value of wins equal to the actual value of wins
- Use $\hat{\theta}_{ij}^W$ as our estimate for the win probability, $\hat{\pi}_i$ for our team i strength estimate, and $\hat{\pi}_j$ for our team j strength estimate
- The maximum likelihood estimate of $\pi_i = e^{\lambda_i}$ satisfies

$$\sum_{j=1}^{t} n_{ij}\frac{\hat{\pi}_i}{\hat{\pi}_i + \hat{\pi}_j} = \sum_{j=1}^{t} n_{ij}\hat{\theta}_{ij}^W = \sum_{j=1}^{t} n_{ij}^W = n_i^W$$

where n_i^W is the total number of wins for one team

Lambda Calculation

- Construct $\hat{\lambda}_i = \ln \hat{\pi}_i$:

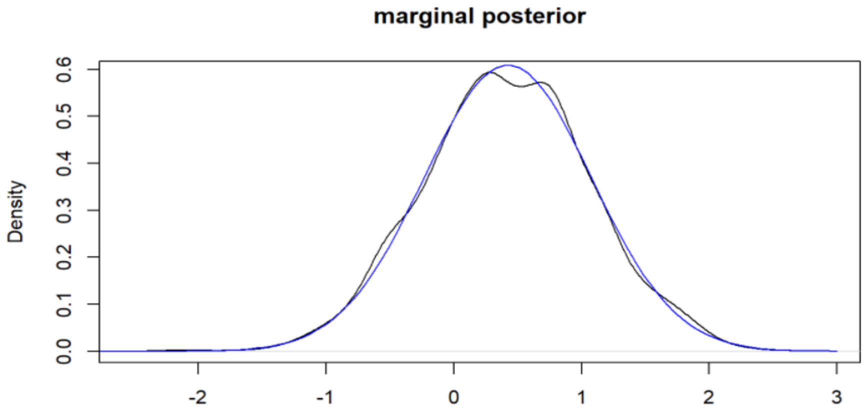
```
##         Cg         Ck         Qn         SL
## -0.5514904  0.3202397  0.7423310 -0.5110803
```

- Note that $\sum_{i=1}^{t} \hat{\lambda}_i = 0$

```
## [1] 3.330669e-16
```

Marginal Posterior

- Estimate the marginal posterior by analyzing the difference in teams strengths between Clarkson and Quinnipiac (black line) and a normal approximation (blue line) with a mean of $\hat{\lambda}_i - \hat{\lambda}_j$ and a variance of $\Sigma_{ii} + \Sigma_{jj} - 2\Sigma_{ij}$



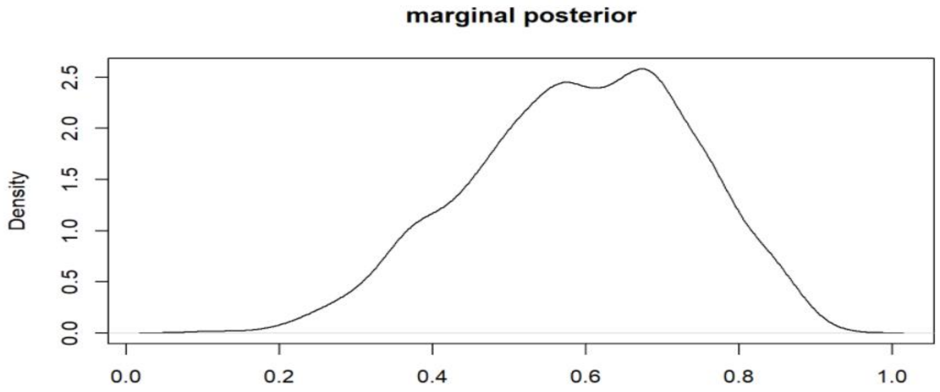
Generating Posterior Samples with Stan

- Using maximum ignorance with regards to our prior, build the model from the exact posterior of the difference among the teams

$$n_{ij}^{W} = \text{binomial\_logit}(n_{ij}, \lambda_i - \lambda_j)$$

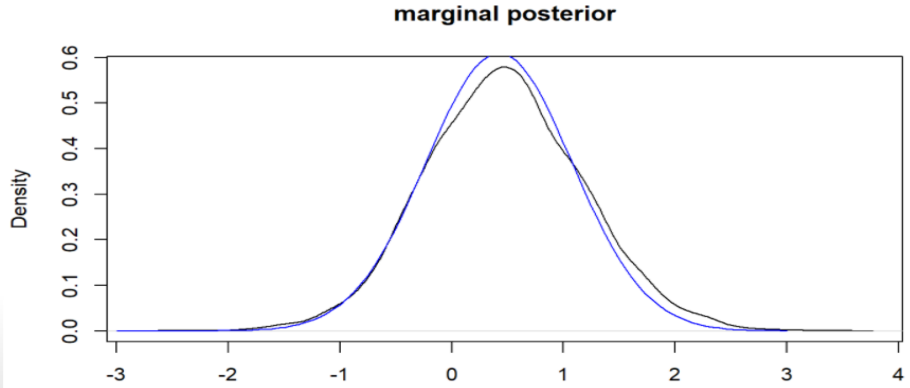
Posterior Probabilities

- Look at the the probability of Quinnipiac beating Clarkson, θ_{ij}, to make sure it lines up with our point estimate denoted by the line



Generating Posterior Samples with Stan (Cont.)

- Same teams used to observe whether the sample distribution lines up with a normal approximation



Bradley-Terry Model for Ties

- Using the premier division of German Hockey, the DEL's 2019-2020 season
- For this section, any games that went to overtime will be counted as ties
- Retain team strengths as the basis, but install a point structure (2 points for a win, 1 point for a tie, 0 points for a loss) instead of just modeling on wins
- Our probabilities of winning, tying, and losing are proportional to π_i, $\nu\sqrt{\pi_i \pi_j}$, and π_j respectively. This means that:

$$(\theta_{ij}^W, \theta_{ij}^T, \theta_{ij}^L) = \text{softmax}(\lambda_i, \tau + \frac{\lambda_i + \lambda_j}{2}, \lambda_j)$$

Bradley-Terry Model for Ties (Cont.)

- Define ν as a function of the probability that a game will go to overtime.
- Calculation of θ_{ij}^W is:

$$\theta_{ij}^W = \frac{\pi_i}{\pi_i + \nu\sqrt{\pi_i \pi_j} + \pi_j}$$

- Similarly, our calculation of θ_{ij}^T is:

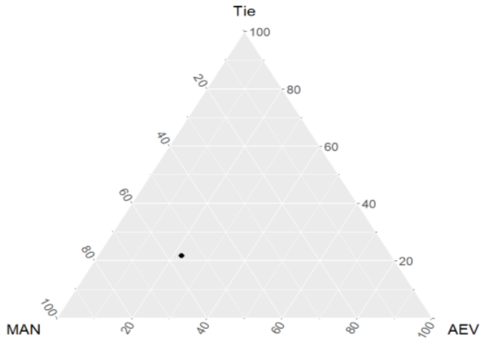
$$\theta_{ij}^T = \frac{\nu\sqrt{\pi_i \pi_j}}{\pi_i + \nu\sqrt{\pi_i \pi_j} + \pi_j}$$

Bradley-Terry Model for Overtime Structure

- Adapt ν to act as a function of the probability of a game going to overtime
- Define three θs for the 3-2-1-0 point structure to obtain a sample distribution: θ_{ij}^{RW} (probability of team i winning in regulation), θ_{ij}^{OW} (probability of team i winning in overtime), and θ_{ij}^{OL} (probability of team i losing in overtime)
- ν iteration will take n^O, the number of games that went to overtime
- Iteration process itself remains same, points and team strengths calculated.

Ternary Plot

- Take two teams, Adler Mannheim (MAN) and Augsburger Panther (AEV), and plot a ternary plot of the MLEs of Mannheim winning, Augsburger winning, and a tie between the two teams:



Full Comparison

- Compare Mannheim and Augsburger by comparing their MLEs for all three game models
- Pure simulation, perhaps teams would be playing differently and chasing different outcomes in different structures
- The simple win-loss model:

```
##    MAN_Wins  AEV_Wins
## 1 0.7192779 0.2807221
```

- The ties model:

```
##    MAN_Wins  AEV_Wins       Tie
## 1 0.5580771 0.2238807 0.2180421
```

- Finally, the overtime model:

```
##   MAN_Wins_Reg MAN_Wins_OT AEV_Wins_OT AEV_Wins_Reg
## 1    0.5126809   0.1925259    0.131485    0.1633082
```